Lecture Notes: Tropical Methods in Moduli Theory and Algebraic Geometry

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Contents

1 Smooth Curves

Let $k = \bar{k}$ be an algebraically closed field. A **curve** is defined as a connected projective reduced algebraic variety of dimension 1 over k.

1.1 Definitions and Basic Properties

Let C be a smooth curve. The Picard group of C is defined as the group of divisors on C modulo linear equivalence

$$\operatorname{Pic}(C) := \frac{\operatorname{Div}(C)}{\sim} \overset{!}{\simeq} \frac{\text{line bundles on } C}{\approx}.$$

For $[D], [D'] \in \operatorname{Pic}(C)$, the group operation is defined by

$$[D] + [D'] = [D + D'].$$

For line bundles $L, L' \in \operatorname{Pic}(C)$, the group operation is defined by their tensor product

$$L \cdot L' = L \otimes L'.$$

The group $\operatorname{Pic}(C)$ is an abelian group. Line bundles are canonically identified with invertible sheaves. There is a natural map

$$D \mapsto \mathcal{O}_C(D).$$

There exists a surjective degree homomorphism

$$\deg\colon \operatorname{Pic}(C) \to \mathbb{Z},$$

defined by

$$[D] \mapsto \deg D,$$

where for a divisor $D = \sum_{P \in C} n_P P$, its degree is $\deg D = \sum_P n_P$.

The kernel of this degree homomorphism is

$$\operatorname{Pic}^0(C) = \{[D] \in \operatorname{Pic}(C) \mid \deg D = 0\},$$

which is a subgroup of $\operatorname{Pic}(C)$. This subgroup is called the **Jacobian** of C, denoted by $\operatorname{Jac}(C) = J_C$.

1.2 Important Line Bundles on Smooth Curves

1. The trivial line bundle $\mathcal{O}_C$ has degree 0.
2. The canonical line bundle K_C is defined as $K_C = T_C^\vee$, where T_C is the tangent line bundle (so K_C is the cotangent line bundle).

For a divisor D, the complete linear system $|D|$ is defined as

$$|D| := \{E \in \operatorname{Div}(C) \mid E \sim D \text{ and } E \geq 0\}.$$

If $|D|$ is nonempty, it forms a projective space $\mathbb{P}^{r(D)}$, where $r(D) = h^0(C, D) - 1$, and

$$\mathbb{P}^{r(D)} = \mathbb{P}(H^0(C, D)).$$

The dimension of the space of global sections of D is

$$h^0(C, D) = \dim H^0(C, D).$$

For the canonical bundle K_C, we have

$$h^0(C, K_C) = \text{genus of } C = g_C, \quad \deg K_C = 2g_C - 2.$$

For the trivial bundle $\mathcal{O}_C$, we have

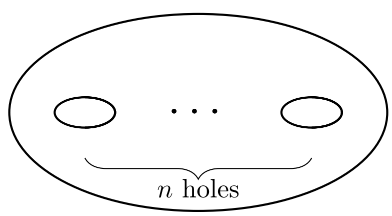
$$H^0(C, \mathcal{O}_C) = k, \quad \deg \mathcal{O}_C = 0.$$

These properties uniquely characterize $\mathcal{O}_C$ (respectively, K_C) in $\mathrm{Pic}(C)$.

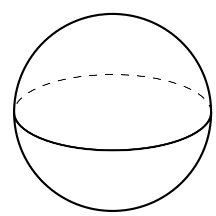
1.3 Special Case: $k = \mathbb{C}$

When $k = \mathbb{C}$, the curve C has an underlying 2-dimensional real manifold S_C, which is a compact, connected, and orientable surface.

Fact 1.1. *The topological genus of S_C is equal to g_C. For a complete proof, see [Mir95, Chapter VI, Section 3, Page 191] .*



genus n complex smooth projective curve



$S_{\mathbb{P}^1} = \mathbb{S}^2$

1.4 Properties of Divisors and Riemann-Roch Theorem

For $D \in \mathrm{Div}(C)$, if $H^0(C, D) \neq 0$, we have a (possibly rational) map

$$\varphi_D \colon C \dashrightarrow \mathbb{P}(H^0(C, D)) = \mathbb{P}^{r(D)=r} = |D|,$$

defined by

$$P \longmapsto (s_0(P) : \cdots : s_r(P)), \quad H^0(C, D) = \langle s_0, \cdots, s_r \rangle.$$

For a hyperplane $H \subset \mathbb{P}^r$, the pullback $\varphi_D^* H \in \mathrm{Div}(C)$ satisfies $\varphi_D^* H \in |D|$. Conversely, for any $E \in |D|$, there exists a hyperplane H_E such that $\varphi_D^* H_E = E$.

Definition 1.2 (Very Ample and Ample Divisors)**.** *A divisor D is*

1. ***very ample*** *if φ_D is an embedding, i.e.,*

$$\varphi_D \colon C \longrightarrow \varphi_D(C)$$

is an isomorphism.

2. ***ample*** *if there exists $n > 0$ such that nD is very ample. This is equivalent to $\deg D > 0$.*

Fact 1.3 (Ample Divisor Criterion)**.** *A divisor D is ample if and only if $\deg D > 0$.*

Fact 1.4 (Very Ample Divisor Criterion)**.** *If $\deg D \geq 2g + 1$, then D is very ample.*

Theorem 1.5 (Riemann-Roch and Serre Duality)**.** *For any $D \in \mathrm{Div}(C)$, the Riemann-Roch theorem states*

$$h^0(C, D) - h^0(C, K_C - D) = \deg D - g + 1,$$

where $h^0(C, D) = \dim H^0(C, D)$.

Corollary 1.6 (Riemann-Roch Corollary)**.** *If $\deg D \geq 2g - 1$, then*

$$h^0(D) = \deg D - g + 1.$$

2 Moduli Spaces of Smooth Curves

Fix the genus g, set-theoretically, the moduli space $\mathcal{M}_g$ is defined as

$$\mathcal{M}_g = \{\text{isomorphism classes of smooth curves of genus } g\}.$$

2.1 Moduli Spaces of Genus 0 and Genus 1 Curves

- **Case** $g = 0$:

  $$\mathcal{M}_0 = \{\text{pt}\}.$$

  The canonical bundle satisfies $\deg K_C = -2$, and the tangent bundle satisfies $\deg T_C = 2$. Thus, T_C is very ample.

  By Riemann-Roch, we have

  $$h^0(C, T_C) = 3.$$

  This implies that the map

  $$\varphi = \varphi_{T_C} : C \hookrightarrow \mathbb{P}^2$$

  is an embedding, and $\varphi(C) \subset \mathbb{P}^2$ is a smooth conic, which is isomorphic to $\mathbb{P}^1$.

- **Case** $g = 1$:

  $$\deg K_C = 0 \Rightarrow K_C = \mathcal{O}_C = T_C.$$

  By Riemann-Roch, we have

  $$h^0(C, K_C) = 1.$$

  There is a bijection

  $$\mathcal{M}_1 \leftrightarrow k = \mathbb{A}^1.$$

  Remark 2.1. *This bijection is actually given by the so-called j-invariant of an elliptic curve. For the case $\mathrm{char}\, k \neq 2$, see [Har77, Chapter IV, section 1]; for the case $\mathrm{char}\, k = 2$, as suggested in [Har77, Chapter IV, section 1], see [Tat74].*

2.2 Genus 2 Curves and Their Moduli Space

2.2.1 Basic Properties

Let C be a smooth curve of genus $g = 2$. The canonical bundle K_C satisfies

$$\deg K_C = 2, \quad h^0(K_C) = 2.$$

The canonical map

$$\varphi = \varphi_{K_C} : C \to \mathbb{P}^1$$

is a degree 2 map. Up to $\mathrm{Aut}(\mathbb{P}^1)$, φ is the unique map of degree 2 to $\mathbb{P}^1$.

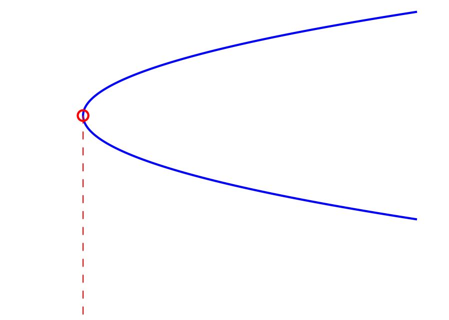


Figure 1: Ramification points for a $2:1$ map

Theorem 2.2 (Riemann-Hurwitz)**.** *Assume* $\operatorname{char} k \neq 2$*. Let* $\varphi\colon C \to D$ *be a map of degree* d *where* C *and* D *are smooth curves. If* $R \in \mathrm{Div}(C)$ *is the ramification divisor of* φ*, then*

$$\deg R = 2g_C - 2 - d(2g_D - 2).$$

In our case, $\deg R = 6$, and φ is ramified at 6 points. Hence, φ is a double cover of $\mathbb{P}^1$ branched at exactly 6 points.

Conversely, given 6 distinct points on $\mathbb{P}^1$, say $\{a_1, \ldots, a_6\}$, where $a_i \in k$ are affine coordinates, there exists a curve C with a degree 2 map

$$\varphi\colon C \to \mathbb{P}^1$$

such that φ ramifies exactly at $\{a_1, \ldots, a_6\}$. By the Riemann-Hurwitz formula, $g_C = 2$.

The function field of $\mathbb{P}^1$ is

$$k(\mathbb{P}^1) = k(x).$$

Consider the extension

$$k(x) \subset k(x)(\sqrt{(x - a_1) \ldots (x - a_6)}).$$

Define the affine curve

$$X = \mathcal{Z}(y^2 - (x - a_1) \ldots (x - a_6)) \subset \mathbb{A}^2.$$

The projective closure $\bar{X} \subset \mathbb{P}^2$ is a singular curve, and C is defined as the normalization (desingularization) of $\bar{X}$.

2.2.2 Moduli Space of Genus 2 Curves

Given 6 distinct points in $\mathbb{P}^1$, say $\{a_1, \dots, a_6\}$, there exists an automorphism of $\mathbb{P}^1$ such that

$$\{a_1, \dots, a_6\} \mapsto \{0, 1, \infty, b_1, b_2, b_3\},$$

where $b_1, b_2, b_3 \in k \setminus \{0, 1\}$.

Define the open affine set

$$U \coloneqq \left((k - \{0, 1\})^3 \setminus \Delta\right),$$

where $\Delta = \Delta_{12} \cup \Delta_{13} \cup \Delta_{23}$, here Δ_{12} is product of the 3rd component and the image of the diagonal map for the 1st and 2nd components, the same for Δ_{13} and Δ_{23}. (Notice that the image of the diagonal map for all the three components Δ_{123} is contained in any Δ_{ij}.)

The map

$$U \to \mathcal{M}_2$$

sends (b_1, b_2, b_3) to the curve constructed above.

- $U \subset \mathbb{A}^3$ is an open affine set.
- The map $U \to \mathcal{M}_2$ is the quotient by a suitable action of the symmetric group S_6.
- The moduli space $\mathcal{M}_2$ is an irreducible quasi-projective (in fact, affine) variety of dimension 3.

2.3 A More Formal Introduction to Moduli Space of Curves

Question: What do we mean by the moduli space of smooth curves of genus g?

Answer: We mean an algebraic variety/scheme $\mathcal{M}_g$ such that

1. There is a bijection between the set of isomorphism classes of smooth curves of genus g over any algebraically closed field k and the set of points in $\mathcal{M}_g(k)$:

$$\mathcal{M}_g(k) \overset{1:1}{\longleftrightarrow} \{\text{Isomorphism classes of smooth curves of genus } g \text{ over } k\}.$$

2. If a family of smooth curves of genus g is given, i.e., a flat morphism $f\colon \mathcal{C} \to B$ where each fiber $\mathcal{C}_b$ is a smooth curve of genus g, then the moduli map

$$\mu_f\colon B \to \mathcal{M}_g$$

is a morphism of schemes. This map sends a point $b \in B$ to the isomorphism class of the fiber $\mathcal{C}_b$:

$$b \mapsto [\mathcal{C}_b].$$

3. The moduli space $\mathcal{M}_g$ is uniquely determined up to isomorphism by the previous two requirements.

4. For any regular map $\varphi\colon B \to \mathcal{M}_g$, there exists a family $f\colon \mathcal{C} \to B$ such that $\mu_f = \varphi$, modulo B-isomorphisms. Here by B-isomorphism we mean for two families $f\colon \mathcal{C} \to B$ and $f'\colon \mathcal{C}' \to B$, there exists an isomorphism $g\colon \mathcal{C} \to \mathcal{C}'$ such that the following diagrams commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow[\simeq]{g} & \mathcal{C}' \\ & {}_{f}\searrow \quad \swarrow_{f'} & \\ & B & \end{array}$$

Remark 2.3. *If $\mathcal{M}_g$ exists and satisfies (1), (2), (3) but not (4), then $\mathcal{M}_g$ is called a **coarse moduli space**; if $\mathcal{M}_g$ exists and satisfies (1), (2), (3), and (4), then $\mathcal{M}_g$ is called a **fine moduli space**.*

Remark 2.4. *(4) contains (1). Indeed, we have*

$$\mathcal{M}_g(k) = \operatorname{Mor}(\operatorname{Spec} k, \mathcal{M}_g) \overset{1:1}{\longleftrightarrow} \textit{Isomorphism classes of genus } g \textit{ smooth curves over } k.$$

and

$$\operatorname{Mor}(B, \mathcal{M}_g) \overset{1:1}{\longleftrightarrow} \textit{families of genus } g \textit{ curves over } B \textit{ up to } B\textit{-isomorphisms.}$$

Theorem 2.5 ((Mumford, 1960s))**.** *$\forall g \geq 1$, there exists a moduli variety/scheme $\mathcal{M}_g$ (defined over $\mathbb{Z}$) which satisfies (1), (2), (3) but NOT (4). Moreover, $\mathcal{M}_g$ is quasi-projective; not projective; integral, of dimension*

$$\dim = \begin{cases} 1, & g = 1 \\ 3g-3, & g \geq 2. \end{cases}$$

Remark 2.6. *There exist*

1. *Examples of maps $\varphi\colon B \to \mathcal{M}_g$ that are not of type μ_f for any family $f\colon \mathcal{C} \to B$.*
2. *Examples where uniqueness in (4) fails.*

Example 2.7 (An isotrivial family)**.** *Assume $\operatorname{char} k \neq 2$. Let $B = \mathbb{A}^1 \setminus \{0\}$. Consider a family $\mathcal{C}$ over B, where $\mathcal{C}_b$ is the normalization of the affine plane curve defined by the equation*

$$by^2 = (x-a_1)\dots(x-a_6), \quad b \in B \subset \overline{B} = \mathbb{A}^1.$$

Then, for $b \in B$, we have $k(\mathcal{C}_b) = k(x)\sqrt{(x-a_1)\dots(x-a_6)}$.

$$f\colon \mathcal{C} \to B, \quad C \coloneqq C_b \simeq C'_b, \quad \forall b, b' \in B.$$

This implies that $\mu_f\colon B \to \mathcal{M}_g$ is constant, because

$$b \mapsto [C], \quad \forall b \in B \textit{ via } \mu_f.$$

Moreover, $f\colon \mathcal{C} \to B$ is isotrivial but not trivial, i.e., it is not a product $C \times B$.

Indeed, let $\overline{f}\colon \overline{\mathcal{C}} \to \overline{B}$ be the extended family. Then $\overline{f}^{-1}(0) \neq C$ as it is a union of six lines. Hence, $\overline{\mathcal{C}} \neq C \times B$.

Remark 2.8.

1. *The fact that $\mathcal{M}_g$ (for $g \geq 1$) is not complete implies that there exist families of smooth curves that are forced to degenerate into singular curves.*
2. *There exists a compactification of $\mathcal{M}_g$ which is itself a moduli space. Specifically, for $g \geq 2$, there exists a projective integral variety $\overline{\mathcal{M}}_g$ such that*
   - *$\mathcal{M}_g$ is open and dense in $\overline{\mathcal{M}}_g$.*
   - *$\overline{\mathcal{M}}_g$ is the moduli space of stable curves of genus g.*
3. *A stable curve of genus $g \geq 2$ is a reduced curve with at most nodes as singularities and finitely many automorphisms.*

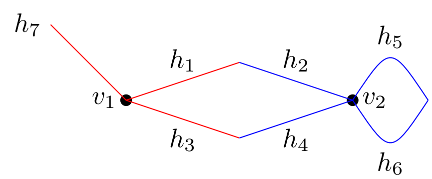
3 Tropical Curves

3.1 A Quick Review on Graph Theory

Definition 3.1. *A graph G with n-legs consists of the following data:*

1. *A finite set of vertices $V = V(G)$.*
2. *A finite set of half-edges $H = H(G)$.*
3. *An involution $\iota\colon H(G) \to H(G)$ with n fixed points, called **legs**. The set of legs is denoted by $L(G)$.*
4. *An endpoint map $\epsilon\colon H(G) \to V(G)$. The **valence** (or **degree**) of a vertex $v \in V(G)$ is defined as $\deg(v) \coloneqq \left|\epsilon^{-1}(v)\right|$.*
5. *A pair $(h, \bar{h})$ of half-edges such that $\iota(h) = \bar{h} \neq h$ is called an **edge**. The set of edges of G is denoted by $E(G) = \left\{\{h, \bar{h}\} \mid h \in H(G)\right\}$ (each edge consists of two half-edges).*

Example 3.2. *Consider the following graph with $V(G) = \{v_1, v_2\}$ and $|H(G)| = 7$:*



The involution $\iota\colon H(G) \to H(G)$ and endpoint map $\epsilon\colon H(G) \to V(G)$ are defined as

$$\begin{aligned} \iota(h_1) &= h_2, & \epsilon(h_1) &= v_1, \\ \iota(h_3) &= h_4, & \epsilon(h_2) &= v_2, \\ \iota(h_5) &= h_6, & &\vdots \\ \iota(h_7) &= h_7, & \epsilon(h_7) &= v_1. \end{aligned}$$

Vertex degrees are

$$\deg(v_1) = 4, \quad \deg(v_2) = 3$$

The set of legs is

$$L(G) = \{h_7\}$$

Definition 3.3. *A morphism from a graph G to another graph G' is a map:*

$$\alpha : V(G) \cup H(G) \to V(G') \cup H(G')$$

satisfying:

$$\alpha(L(G)) \subseteq L(G') \cup V(G')$$

and making the following diagrams commute

$$\begin{array}{ccc} V(G)\cup H(G) & \xrightarrow{\alpha} & V(G')\cup H(G') \\ {\scriptstyle (\mathrm{id},\epsilon)}\downarrow & & \downarrow{\scriptstyle (\mathrm{id},\epsilon')} \\ V(G)\cup H(G) & \xrightarrow{\alpha} & V(G')\cup H(G') \end{array}$$

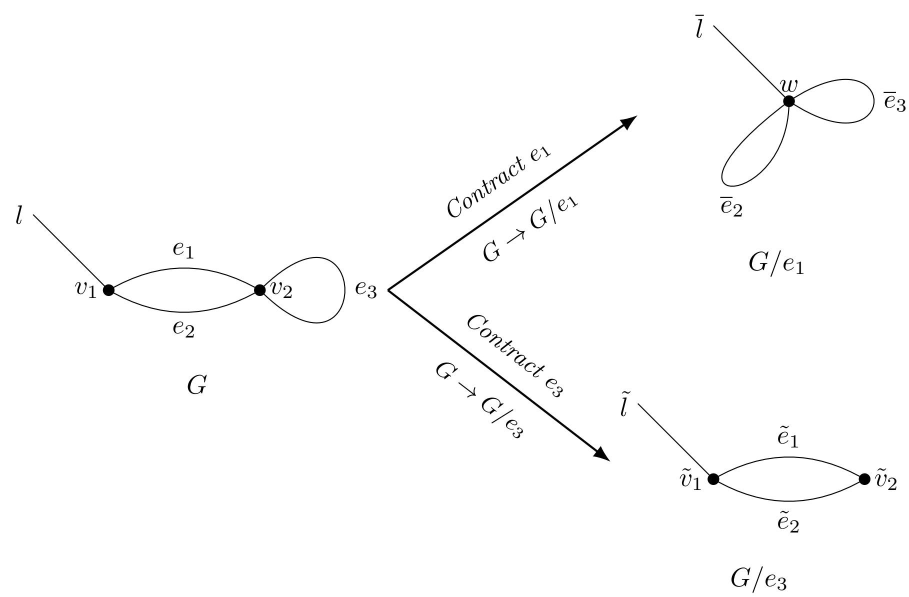
$$\begin{array}{ccc} V(G)\cup H(G) & \xrightarrow{\alpha} & V(G')\cup H(G') \\ {\scriptstyle (\mathrm{id},\iota)}\downarrow & & \downarrow{\scriptstyle (\mathrm{id},\iota')} \\ V(G)\cup H(G) & \xrightarrow{\alpha} & V(G')\cup H(G') \end{array}$$

Note: *The first diagram implies $\alpha(V(G)) \subseteq V(G')$, and we have $\alpha(E(G)) \subseteq V(G') \cup E(G')$.*

Example 3.4 (Contraction of an edge)**.** *Let $e \in E(G)$ be an edge with distinct endpoints v and w. The contraction G/e is defined by*

$$\begin{aligned} E(G/e) &= E(G) \setminus \{e\}, \\ V(G/e) &= (V(G) \setminus \{v, w\}) \cup \{\overline{v}\}, \\ L(G/e) &= L(G). \end{aligned}$$

For the following graph G with $E(G) = \{e_1, e_2, e_3\}$ and $L(G) = \{f\}$, the contractions G/e_1 and G/e_3 are illustrated below:



Remark 3.5.

- *The genus, or the first Betti number, of a connected graph is defined as*

$$g(G) \coloneqq \mathrm{rk}_{\mathbb{Z}} H_1(G, \mathbb{Z}) = \#E - \#V + 1.$$

- *For n non-connected graphs G,*

$$g(G) := \operatorname{rk}_{\mathbb{Z}} H_1(G, \mathbb{Z}) = \#E - \#V + c,$$

where c is the number of connected components.

- *Intuitively, $g(G)$ corresponds to the "number of holes" in the graph. As we have seen, contraction of an edge does not preserve genus.*

  Indeed, in Example 3.4, we have $g(G) = 2$, $g(G/e_1) = 2$ and $g(G/e_3) = 1$.

- *Let $e \in E(G)$. If one of its endpoints has degree 1, then e is a leaf (or leaf edge).*

3.2 Basics of Tropical Curves

Definition 3.6. *An abstract (pure) tropical curve is a pair $\Gamma = (G, \ell)$, where*

- *G is a graph with no legs ($L(G) = \emptyset$);*
- *$\ell\colon E(G) \to \mathbb{R}_{>0} \cup \{\infty\}$ is a function such that*

$$\ell(e) = \infty \iff e \text{ is a leaf.}$$

The original definition by Mikhalkin does not include the term "pure".

Notation.

- For convenience, when the graph G is clear, we will denote $E := E(G)$, $L := L(G)$ and $V := V(G)$ for short.
- Unless otherwise stated, $V \neq \emptyset$, G is connected.
- $g(\Gamma) = g(G) = |E| - |V| + 1$.

Definition 3.7. *For any $n \geq 0$, a pure n-pointed/marked tropical curve is a pair $\Gamma = (G, \ell)$, where*

- *G is a graph with $|L| = n$;*
- *$\ell\colon E \cup L \to \mathbb{R}_{>0} \cup \{\infty\}$ is a function such that*

$$\ell(x) = \infty \iff x \in L \text{ or } x \text{ is a leaf edge.}$$

Definition 3.8. *Let $\Gamma = (G, \ell)$ and $\Gamma' = (G', \ell')$ be two n-pointed pure tropical curves of genus g. If they can be obtained from one to the other, up to isomorphism, by a finite sequence of the following operations (see Figure 2 below): adding or removing*

1. *leaf edge and the adjacent vertex;*
2. *vertex of degree 2;*

then we say Γ and Γ' are equivalent.

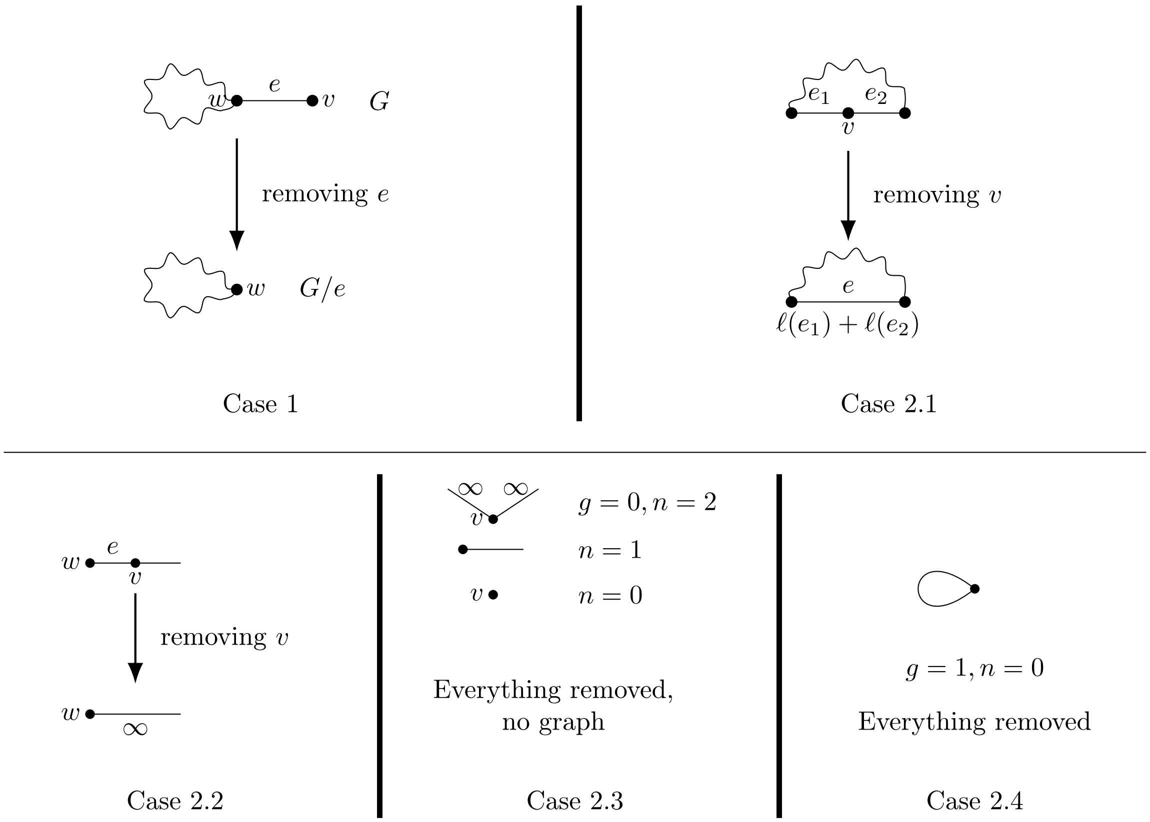


Figure 2: Leaf Edge and deg 2 Vertex Contraction Cases

Proposition 3.9. *Let $g, n \geq 0$ be such that $2g - 2 + n \geq 1$. Then every equivalence class of pure tropical curve has a unique representative (up to isomorphism) having all vertices of* deg $\geq$ 3*. This representative is called STABLE or CANONICAL, denoted as $\Gamma = (G, \ell) \to \Gamma^{stable} = (G^{stable}, \ell^{stable})$.*

Proof.

Step 1. Remove all vertices of degree 1 and the adjacent edge.

Step 2. Remove all vertices of degree 2.

□

Example 3.10. *(Stable graphs with $g = 0$ and $n = 4$) The following are the only four stable graphs with $g = 0$ and $n = 4$. Note that with the legs marked, they are pairwise non-isomorphic.*

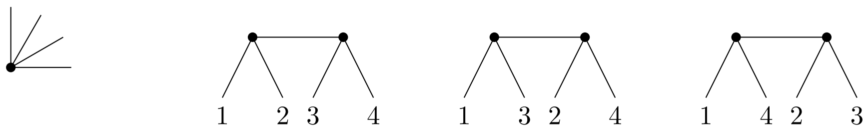


Figure 3: Stable graphs with $g = 0, n = 4$

Example 3.11. *The following is an example of the contraction procedure described in Proposition 3.9:*

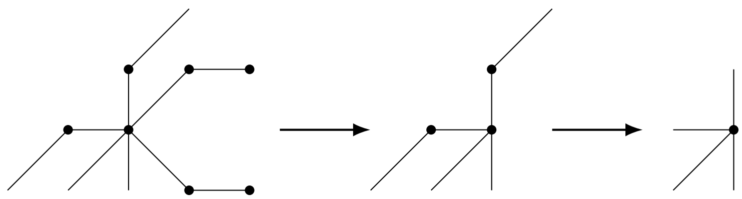


Figure 4: An example of contraction to stable graph

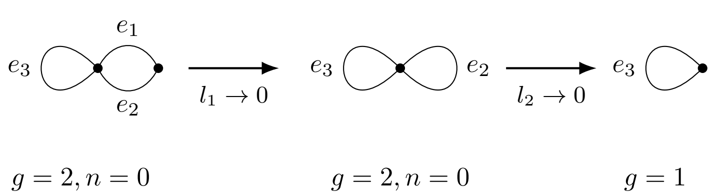
Remark 3.12. *If Γ is stable, then $\Gamma \backslash L$ is a metric space by defining the distance between two points to be the minimal sum of path lengths. So (stable pure) tropical curves are sometimes called metric graphs.*

Definition 3.13. *Fix a stable graph G, let $\delta \coloneqq |E(G)|$, and $E(G) = \{e_1, \cdots, e_\delta\}$, for any $\underline{l} = (l_1, \cdots, l_\delta) \in \mathbb{R}^\delta_{>0}$, there exists a unique function*

$$\ell \colon E(G) \to \mathbb{R}^\delta_{>0} \quad \text{such that} \quad \ell(e_i) = l_i, \forall i.$$

Then $\Gamma \coloneqq (G, \ell)$ is called the tropical curve corresponding to $\underline{l}$.

Example 3.14. *In the following example, $l_i = \ell(e_i)$.*



Definition 3.15. *(weighted tropical curve)([BMV11, Definition 3.1.1 and Definition 3.1.3])*

A (weighted) graph with n-legs is a pair (G, w) where

- *G is a graph, $G = (V, E, L)$, $|L| = n$;*
- *$w \colon V \to \mathbb{Z}_{>0}$ is the weight function.*

Its genus is defined to be

$$g(G,w) = b_1(G) + \sum_{v\in V} w(v) = g(G) + \text{weights of vertices.}$$

A (weighted) tropical curve with n*-legs is a triple* $\Gamma = (G, w, \ell)$ *where*

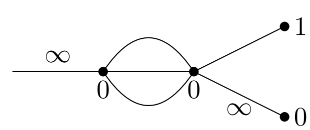
- (G, w) *is a weighted graph;*
- $\ell\colon E \cup L \to \mathbb{R}_{>0} \cup \{\infty\}$ *is a function such that*

$$\ell(x) = \infty \iff x \in L \text{ or } x \in E \text{ is a leaf where leaf vertex has weight zero.}$$

Let Γ *and* Γ' *be two weighted tropical curves, then we say they are equivalent if they can be obtained from one to the other by a finite sequence of the following operations:*
adding or removing

1. *leaf vertices of weight zero and the adjacent edges;*
2. *vertices of* $\deg 2$ *and weight* 0*.*

Example 3.16. *The following is an example of weighted tropical curve of genus* 3 *with one leg:*



Same as in Proposition 3.9, we have

Proposition 3.17. *Let* $g, n \geq 0$ *be such that* $2g-2+n \geq 1$*. Then every equivalence class of weighted tropical curve has a unique representative (up to isomorphism) having all vertices of* $\deg \geq 3$*. This representative is called STABLE or CANONICAL.*

Definition 3.18. *Let* $\Gamma = (G, w, \ell)$ *and* $\Gamma' = (G', w', \ell')$ *be two (weighted) tropical curves. We say* Γ *is isomorphic to* Γ'*, denoted as* $\Gamma \simeq \Gamma'$*, if there is an isomorphism of graphs* $\alpha\colon G \to G'$*, i.e.,*

$$\alpha_V\colon V(G) \xrightarrow{1:1} V(G'), \quad \alpha_E\colon E(G) \xrightarrow{1:1} E(G'), \quad \alpha_L\colon L(G) \xrightarrow{1:1} L(G')$$

such that

$$w'(\alpha_V(v)) = w(v), \forall v \in V; \quad \ell'(\alpha_E(e)) = \ell(e), \forall e \in E.$$

Remark 3.19. *For any given* g, n*, there are finitely many stable graphs (up to isomorphism).*

3.3 Moduli Space of Tropical Curves

From now on, unless otherwise stated, graphs and tropical curves will be weighted graphs and weighted tropical curves as in Definition 3.15.

Let

$$\mathcal{G}_{g,n} \coloneqq \{\text{stable weighted graphs of genus } g \text{ with } n\text{-legs }\}/\simeq,$$

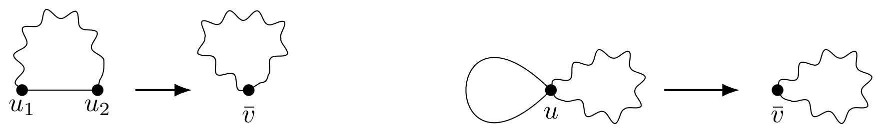
then $\mathcal{G}_{g,n}$ is a finite, and it is empty if $2g-2+n < 1$.

We want to make $\mathcal{G}_{g,n}$ into a poset. To do this, we need the following operations:

Definition 3.20 (Weighted Contraction of Edges)**.** *Let $(G = (V, E, L), w\colon V \to \mathbb{Z})$ be any graph, for an edge $e \in E$, we define its weighted contraction $(G/e, w/e)$ to be*

- *G/e is defined as in Example 3.4 with the contraction morphism $\sigma\colon G \to G/e$;*
- $$w/e(\bar{u}) = \begin{cases} w(u_1) + w(u_2), & \text{if } e \text{ is not a loop, } \sigma(e) = \bar{u} \text{ is a point, and } u_1, u_2 \text{ are the two endpoints of } e; \\ w(e) + 1, & \text{if } e \text{ is a loop, } \sigma(e) = \bar{u} \text{ is a point;} \\ w(u), & \text{if } e \neq u \text{ and } \sigma(u) = \bar{u}. \end{cases}$$

We also denote this weighted contraction as $\sigma\colon (G, w) \to (G/e, w/e)$.



$w/e(\bar{v}) = w(u_1) + w(u_2)$ $\qquad$ $w/e(\bar{v}) = w(u) + 1 = w(u) + g\left(\sigma^{-1}(\bar{v})\right)$

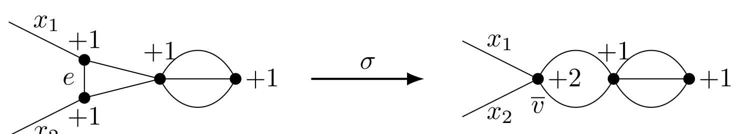
More generally, for any set $S \subset E$ (with empty set taken into consideration), we can define its weighted contraction $\sigma\colon (G, w) \to (G/S, w/S)$ as follows:

- *(G/s) is obtained by applying the construction of contracting one edge repeatedly, and we denote this contraction of graph by $\sigma\colon G \to G/S$;*
- *$E(G/S) = E \backslash S,\ V(G/S) = (V \backslash \{$ endpoints of edges in S $\}) \bigcup (\bigcup_{s \in S} \{\sigma(s)\}),\ L(G/S) = L(G)$;*
- *For any $\bar{u} \in V(G/S)$, we have*

$$w/S(\bar{u}) := b_1(\sigma^{-1}(\bar{u})) + \sum_{v \in \sigma^{-1}(\bar{u})} w(v) = g(\sigma^{-1}(v)),$$

where $\sigma^{-1}(\bar{u})$ is the subgraph of G that is spanned by all edges mapping to $\bar{u}$.

Example 3.21. *Here is an example of weighted contraction of an edge in a graph of genus 7:*



G $\qquad$ $G/e,\ \bar{v} = \sigma(e)$

Remark 3.22. *1. $g(G) = g(G/S)$.*

Claim 3.23. *Proof.* If e is a not a loop, then we have

- $|E(G)| = |E(G/e)| + 1$;
- $|V(G)| = |V(G/e)| + 1$;

- $\sum_{v\in V(G)} w(v) = \sum_{\bar{v}\in V(G/e)} w/e(\bar{v})$.

We obtain

$$g(G/e) = |E(G/e)| - |V(G/e)| + 1 + \sum_{\bar{v}\in V(G/e)} w/e(\bar{v}) = |E(G/e)| + 1 - (|V(G/e)| + 1) + 1 + \sum_{v\in V(G)} w(v) = g(G).$$

If e is a loop, then we have

- $|E(G)| = |E(G/e)| + 1$;
- $|V(G)| = |V(G/e)|$;
- $\sum_{v\in V(G)} w(v) = \sum_{\bar{v}\in V(G/e)} w/e(\bar{v}) - 1$.

We obtain

$$g(G/e) = |E(G/e)| - |V(G/e)| + 1 + \sum_{\bar{v}\in V(G/e)} w/e(\bar{v}) = |E(G/e)| + 1 - |V(G/e)| + 1 + \sum_{v\in V(G)} w(v) - 1 = g(G).$$

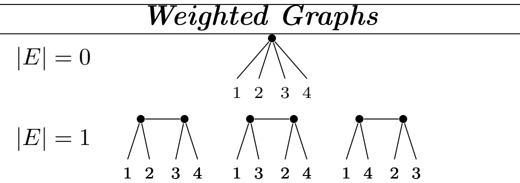
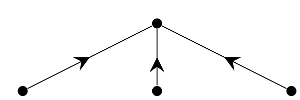
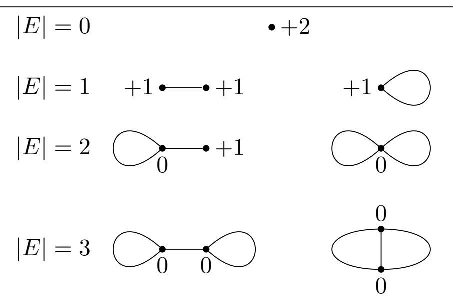
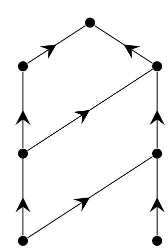
□

2. *If $G \in \mathcal{G}_{g,n}$, then $G/S \in \mathcal{G}_{g,n}$.*

   Here we only care about weight 0 vertices. Let v be a vertex with $w(v) = 0$. If $\bar{v} := \sigma(v)$ has weight 0, then its degree will not drop; if $\bar{v} := \sigma(v)$ has weight > 0 (possibly by contracting a loop), then we do not care about it any more.

Definition 3.24. *Let G, G' be two stable weighted graphs in $\mathcal{G}_{g,n}$, then we say $G \geq G'$ if G' is a weighted contraction of G.*

Example 3.25. *Here we draw the poset structure given by weighted contraction for $\mathcal{G}_{0,4}$ and $\mathcal{G}_{2,0}$.*

<table>
<tr><th></th><th>Weighted Graphs</th><th>Poset Structure</th></tr>
<tr><td>$\mathcal{G}_{0,4}$</td><td>
</td><td></td></tr>
<tr><td>$\mathcal{G}_{2,0}$</td><td>
</td><td></td></tr>
</table>

Lemma 3.26. *Let $G \in \mathcal{G}_{g,n}$. Then*

$$|E(G)| \leq 3g - 3 + n,$$

and the following are equivalent:

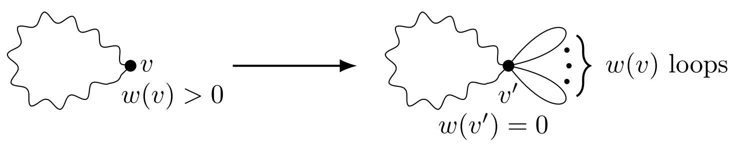
(a) $|E(G)| = 3g - 3 + n$*;*

(b) $\forall v \in V(G)$*, we have* $w(v) = 0$ *and* $\deg v = 3$*;*

(c) $|V(G)| = 2g - 2 + n$ *and* $w(v) = 0$ *for any* $v \in V(G)$*.*

If the equivalent conditions are satisfied, we say G is regular/3-regular.

Proof. Suppose $\exists v \in V$ such that $w(v) > 0$. Then consider the graph G' obtained from G by replacing v by a vertex v' of weight 0 and $w(v)$ loops attached to it.



Then G' is contained in $\mathcal{G}_{g,n}$ as well, and $|E(G')| \geq |E(G)|$. Therefore, if $|E(G)|$ is maximum, we must have $w \equiv 0$, and in this case, we must have $\deg v \geq 3$ for any $v \in V(G)$ by stability of G'.

Then, we have

$$|E(G')| \geq \frac{1}{2}\Big(\sum_{v \in V(G')} \deg v - n\Big) \geq \frac{1}{2}\left(3|V(G)| - n\right),$$

while $g = g(G) = g(G') = |E(G')| - |V(G')| + 1$.

Therefore, we have

$$|E(G')| = g + |V(G')| - 1 \leq g - 1 + \frac{2}{3}|E(G')| + \frac{n}{3},$$

from this we deduce

$$|E(G)| \leq |E(G)'| \leq 3g - 3 + n. \tag{1}$$

Now we show that (a) implies (b) and (c), the other cases are left as exercises.

Suppose $|E(G)| = 3g - 3 + n$, this implies $w \equiv 0$, hence $\deg v \geq 3$ for any $v \in V(G)$. Therefore, in equation (1), the equality hols, and equality holds everywhere before. Hence, $\deg v = 3$ for any $v \in V(G)$, this proves (b).

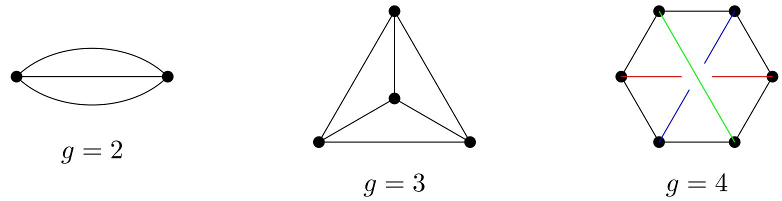
And

$$|V(G)| = \frac{2}{3}|E(G)| + \frac{n}{3} = \frac{2}{3}(3g - 3 + n) + \frac{n}{3} = 2g - 2 + n$$

gives (c).

□

Example 3.27. *Polygons in $2g-2$ vertices with all $(g-1)$ diagonals added are examples of 3-regular graph for $n = 0, \forall g$.*



Exercise 3.28. *Draw the poset for $\mathcal{G}_{3,0}$.*

Now we are ready to define the moduli space os tropical curves.

Definition 3.29. *Set theoretically, we have*

$$M_{g,n}^{trop} := \{\text{equivalent classes of } n\text{-marked tropical curves of genus } g\},$$

and

$$M_G^{trop} := \{\Gamma = (G, \ell), \forall \ell\}/\sim,$$

where $G = (V, E, L, w)$ is a stable graph.

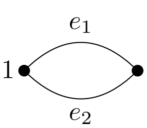
Clearly from the definition, we have

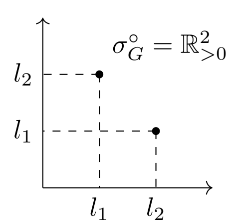
$$M_{g,n}^{trop} = \bigsqcup_{G \in \mathcal{G}_{g,n}} M_G^{trop}.$$

Fix $G \in \mathcal{G}_{g,n}$, denote its edges as $E = \{e_1, \cdots, e_\delta\}$, where $\delta = |E|$. Then we define

$$\sigma_G^\circ := \mathbb{R}_{>0}^\delta$$

which is an open cone with Euclidean topology. Recall that for any $\underline{l} = (l_1, \cdots, l_\delta) \in \sigma_G^\circ$, we can associate to it a tropical curve $\Gamma_{\underline{\ell}} = (G, \ell_{\underline{l}})$, where $\ell_{\underline{l}}(e_i) = l_i$ (see Definition 3.13). It can happen that $\underline{l}_1 \neq \underline{l}_2$, but $\Gamma_{\underline{l}_1} \simeq \Gamma_{\underline{l}_2}$. Therefore, $\mathrm{Aut}(G)$ acts on σ_G° by permuting coordinates in a suitable way.

Example 3.30. *Let $G =$ $+1$  $+1$, then $g = 3$, and $\sigma_G^\circ = \mathbb{R}_{>0}^2$. For any $l_1, l_2 \in \mathbb{R}_{>0}, l_1 \neq l_2$, we have $\Gamma_{(l_1,l_2)} \simeq \Gamma_{(l_2,l_1)}$.*



Claim 3.31. *$\sigma_G^\circ / \operatorname{Aut}(G) = M_G^{trop}$.*

Here, $\sigma_G^\circ = M_G^{trop}$ is a topological space with the quotient topology induced by the Euclidean topology on σ_G°.

Remark 3.32. *It can happen that $\operatorname{Aut}(G) \neq \{e\}$, but the action is trivial. Consider, for example, the graph* $+1 \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet +1$.

Now we denote

$$\sigma_G \coloneqq \mathbb{R}_{\geq 0}^{\delta}$$

which is a closed cone with Euclidean topology and contains σ_G° as an open dense subset.

For any $\underline{l} \in \sigma_G$, we associate to it a subset of $E(G)$ $S_{\underline{l}} \coloneqq \{e_i \mid l_i = 0\}$. Then, we can associate to $\underline{l}$ a tropical curve $\Gamma_{\underline{l}} \coloneqq (G/S_{\underline{l}}, \ell_{\underline{l}})$.

Recall that via the weighted contraction of edges, $\mathcal{G}_{g,n}$ is equipped with the structure of a poset (see Definition 3.20, Remark 3.22, and Definition 3.24). In this way, σ_G parametrized "rigidified" tropical curves of type $\Gamma' = (G', \ell')$ such that $G' \in \mathcal{G}_{g,n}$ and $G' \leq G$.

Let $\varphi_G \colon \sigma_G^\circ \to M_{g,n}^{trop}$ be the composition of the quotient map $\sigma_G^\circ \twoheadrightarrow M_G^{trop}$ followed and the inclusion $M_G^{trop} \hookrightarrow M_{g,n}^{trop}$. It lifts to a map $\bar{\varphi}_G \colon \sigma_G \to M_{g,n}^{trop}$ via the partial order given by weighted contraction. For a weighted graph $G' \leq G$, $\operatorname{Aut}(G)$ act on G' via the contraction $G \to G'$, this gives the map $\sigma_G \to \sigma_G / \operatorname{Aut}(G)$. By construction, $\bar{\varphi}_G$ factors through $\sigma_G / \operatorname{Aut}(G)$. All these can be summarized to the following commutative diagram:

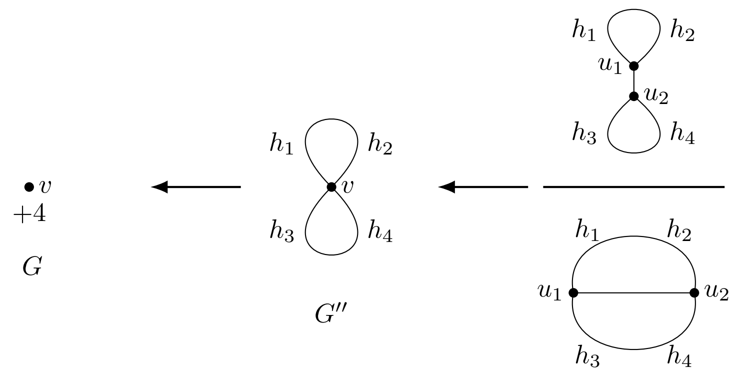
$$\begin{array}{ccc} \sigma_G^\circ & \xrightarrow{\varphi_G} & M_{g,n}^{trop} \\ \downarrow & \overset{\bar{\varphi}_G}{\nearrow} & \uparrow \\ \sigma_G & \longrightarrow & \sigma_G / \operatorname{Aut}(G) \end{array}$$

Here the map $\sigma_G^\circ \to \sigma_G$ is the inclusion. Note that $\sigma_G / \operatorname{Aut}(G) \to M_{g,n}^{trop}$ is not a quotient map .

Proposition 3.33. *Let $G \in \mathcal{G}_{g,n}$, then there exists $G' \in \mathcal{G}_{g,n}$ such that $E(G') = 3g - 3 + n$ and $G' \geq G$. In particular, we have $M_G^{trop} \subset \overline{M_{G'}^{trop}}$.*

Proof. Assume $|E(G)| < 3g - 3 + n$. For a vertex v with $\deg v = d \geq 4$, denote the set of half edges attached to v as $H_v = \{h_1, \cdots, h_d\}$. Replace v by an edge whose two end points are denoted as u_1, u_2 such that $H_{u_1} = \{h_1, \cdots, h_{\lfloor d/2 \rfloor}\}$ and $H_{u_2} = \{h_{\lfloor d/2 \rfloor + 1}, \cdots, h_d\}$.

In the rest of the proof, we use the example of a single point of weight 4, as in the following picture, to illustrate in the idea of the proof:



Define the involution map accordingly to H_{u_1} and H_{u_2}, and notice that for any $i = 1, 2$,

$$\deg u_i = \begin{cases} \frac{d}{2} + 1, & d \text{ even} ; \\ \frac{d+1}{2}, & d \text{ odd} . \end{cases}$$

And $\deg u_i < d$ in both cases. By induction, we are done. □

From this, we immediately obtain

Proposition 3.34. *The map*

$$\bar{\varphi}\colon \bigcup_{\substack{G \in \mathcal{G}_{g,n} \\ |E|=3g-3n}} \sigma_G \longrightarrow M_{g,n}^{trop}$$

is surjective.

Proof. Indeed, from Proposition 3.33, we see that for any $G \in \mathcal{G}_{g,n}$, there exists $G' \in \mathcal{G}_{g,n}$ with $|E(G')| = 3g - 3 + n$ such that $\sigma_{G'} \supset \sigma_G^\circ$, as $M_{g,n}^{trop} = \bigsqcup_{G \in \mathcal{G}_{g,n}} M_G^{trop}$ and $M_G^{trop} = \varphi_G(\sigma_G^\circ) \subset \bar{\varphi}_G(\sigma_G)$, the statement immediately follows. □

We end this chapter by mentioning the following theorem:

Theorem 3.35. *Let $2g - 2 + n > 0$. Consider $M_{g,n}^{trop} = \bigsqcup_{G \in \mathcal{G}_{g,n}} M_G^{trop}$, then:*

1. *$M_{g,n}^{trop}$ is a connected Hausdorff topological space of pure dimension $3g - 3 + n$;*
2. *$\forall G \in \mathcal{G}_{g,n}$, $\overline{M_G^{trop}} = \bigcup_{G' \leq G} M_{G'}^{trop}$;*
3. *$M_{g,n}^{reg} := \bigcup_{G \text{ regular}} M_G^{trop}$ is open and dense in $M_{g,n}^{trop}$;*
4. *$M_{g,n}^{pure} := \bigcup_{(G,w), w \equiv 0} M_G^{trop}$ is open and dense in $M_{g,n}^{trop}$.*

4 Moduli Spaces of Stable Curves

A standard reference for the geometry of nodal curves and stable curves is [ACG11, Chapter X].

4.1 Nodal Curves, Stable Curves and Their Dual Graphs

Recall that for any $g \geq 2$, $\mathcal{M}_g$ is a non-projective variety of dim $= 3g - 3$ (Theorem 2.5), and we have seen what a stable curve is in Remark 2.8, but let us formulate it again here:

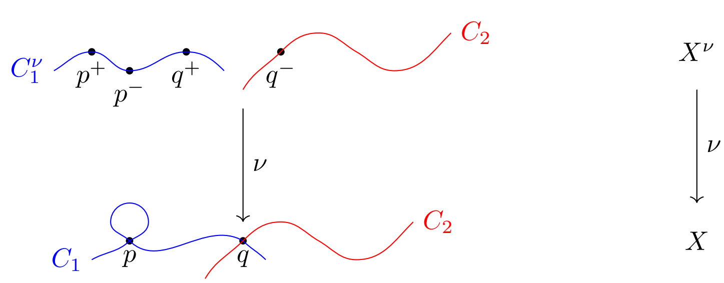
Definition 4.1. *A nodal curve is a reduced projective curve X such that X has at most nodes as singularities.*

A stable curve is a reduced connected projective curve X such that X has at most nodes as singularities and $\mathrm{Aut}(X)$ is finite, that is, a connected nodal curve with finite automorphism group.

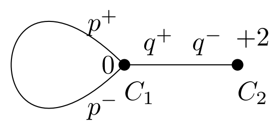
Remark 4.2.
1. *Recall that a point $p \in X$ is a node if $\hat{\mathcal{O}}_{p,X} \simeq k[[x,y]]/(x,y)$. In other words, in the formal neighbourhood of p (or, analytically locally around p), X looks like the union of two different lines.*
2. *The normalization map of X is denoted as $\nu\colon X^\nu \to X$. If $X = \cup_{i=1}^r C_i$, where C_i's are the irreducible components of X, then X^ν is the disjoint union of C_i^ν's, i.e., $X^\nu = \amalg_i^r C_i^\nu$.*
3. *The set of singular points of X is denoted as X_{sing}, that is,*
$$X_{sing} = \{\text{nodes of } X\}.$$
For any $p \in X$, we have $\nu^{-1}(p) = \{p^+, p^-\}$.
4. *We can recover X from X^ν and X_{sing}, that is,*
$$X = X^\nu / \{p^+ \sim p^-, \forall p \in X_{sing}\}.$$

Definition 4.3. *To a nodal curve X, we define its dual graph (G_X, w_X) as follows:*
1. *Its set of vertices is $V_X \coloneqq \{\text{irreducible components of } X\} = \{C_1, \cdots, C_r\}$.*
2. *Its set of half edges is $H_X \coloneqq \{p^+, p^-; \forall p \in X_{sing}\}$.*
3. *The involution map $\iota_X\colon H_X \to H_X$ is given by $\iota_X(p^+) = p^-$.*
4. *Its set of edges is $E_X \coloneqq \{\{p^+, p^-\}, \forall p \in X_{sing}\} = X_{sing}$.*
5. *The end point map $\epsilon_X\colon H_X \to V_X$ is given by $\epsilon_X(q) = C_i$ if $q \in C_i^\nu$.*
6. *The weight function $w_X\colon V_X \to \mathbb{Z}_{\geq 0}$ is given by $w_X(C_i) = g(C_i^\nu)$.*

Example 4.4. *Let $X = C_1 \cup C_2$ be a stable curve of genus 3, where C_1 is an irreducible nodal curve of arithmetic genus 1 with a unique node p (which implies that C_1^ν is a smooth rational curve), C_2 is a smooth curve of genus 2 and $C_1 \cap C_2 = \{q\}$. Its normalization is $X^\nu = C_1^\nu \sqcup C_2^\nu$.*



The dual graph of X is



For a nodal curve X, we have

$$\mathrm{Pic}(X) = \frac{\text{Cartier divisors}}{\sim} = \frac{\text{line bundles}}{\simeq}.$$

Similar to the cases for smooth curves, a nodal curve has two special invertible sheaves:

- The structural sheaf (corresponding to the trivial line bundle) $\mathcal{O}_X$;
- the dualizing sheaf (corresponding to the so-called dualizing line bundle) ω_X.

The dualizing sheaf provides the Serre duality, which gives

$$H^1(X, L) \simeq H^0(X, \omega_X \otimes L^{-1})^\vee, \quad h^1(X, L) = h^0(X, \omega_X \otimes L^{-1}), \quad \forall L \in \mathrm{Pic}(X).$$

For the construction and more properties of the dualizing sheaf of a projective scheme over a field, we refer to [Har77, Chapter III, Section 7].

Definition 4.5. *The (arithmetic) genus of a nodal curve X is*

$$g_X \coloneqq h^0(X, \omega_X) = h^1(X, \mathcal{O}_X).$$

Remark 4.6. *Let X be a nodal curve. For any irreducible component $C \in V_X$, we have*

$$\deg C = 2|C_{sing}| + |C \cap \overline{X \setminus C}|.$$

Lemma 4.7. *For a connected nodal curve X, we have*

$$g_X = g(G_X, w_X).$$

Proof. We have a short exact sequence

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{X^\nu} \to \mathcal{S} \to 0,$$

where $\mathcal{S}$ is a skyscraper sheaf supported exactly on X_{sing}, that is, $\mathcal{S}_p = k$ for any $p \in X_{sing}$ and $\mathcal{S}_q = 0$ for any $q \notin X_{sing}$.

Take the long exact sequence associated to it, we obtain

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \nu_* \mathcal{O}_{X^\nu}) \to H^0(X, \mathcal{S}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \nu_* \mathcal{O}_{X^\nu}) \to 0.$$

Therefore, we have

$$g_X = h^1(X, \mathcal{O}_X) = h^1(X, \nu_* \mathcal{O}_{X^\nu}) + h^0(X, \mathcal{S}) - h^0(X, \nu_* \mathcal{O}_{X^\nu}) + h^0(X, \mathcal{O}_X).$$

Here we have (check Remark 4.8 below):

- $h^1(X, \nu_* \mathcal{O}_{X^\nu}) = h^1(X^\nu, \mathcal{O}_{X^\nu}) = \sum_{i=1}^r h^1(C_i^\nu, \mathcal{O}_{C_i^\nu}) = \sum_{i=1}^r g_{C_i^\nu} = \sum_{i=1}^r w(C_i)$;
- $H^0(X, \mathcal{S}) = k^{\oplus |X_{sing}|}$ and $h^0(X, \mathcal{S}) = |X_{sing}| = |E_X|$;
- $h^0(X, \nu_* \mathcal{O}_{X^\nu}) = h^0(X^\nu, \mathcal{O}_{X^\nu}) = \#$ of connected components $= r = |V_X|$;
- $h^0(X, \mathcal{O}_X) = 1$.

Combining all these, we obtain

$$g_X = \sum_{i=1}^r w(C_i) + |E_X| - |V_X| + 1 = g(G_X, w_X).$$

□

Remark 4.8. *Here we use the fact that $\nu \colon X^\nu \to X$ is a finite morphism (see [Har77, Chapter II, Proposition 6.8, Page 137 and Chapter III, Exercise 11.2, Page 280]). In particular, ν is an affine morphism. Therefore, for any quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X^\nu)$, we have*

$$H^i(X^\nu, \mathcal{F}) \simeq H^i(X, \nu_* \mathcal{F}), \quad \forall i \in \mathbb{Z}.$$

(See [Har77, Chapter III, Exercise 8.2, Page 252].)

Remark 4.9. *For a nodal curve $X = \bigcup_{i=1}^\gamma C_i$ with γ irreducible components, from Lemma 4.7, we immediately obtain*

$$g_X = |X_{sing}| - \gamma + 1 + \sum_{i=1}^{\gamma} g_{C_i^\nu} = |E_{G_X}| - |V_{G_X}| + 1 + \sum_{i=1}^{\gamma} w_X(v_i).$$

Lemma 4.10. *Let X be a nodal curve. Then X is stable if and only if $g_X \geq 2$ and the following equivalent conditions hold:*

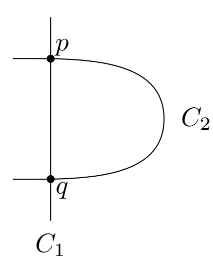
1. *For any component $C \subset X$ such that $C^\nu \simeq \mathbb{P}^1$, we have*

$$\deg C = |C \cap (\overline{X \setminus C})| + 2|C_{sing}| \geq 3;$$

2. *The dual graph of X is stable.*

Proof. It is clear from the definition that the two conditions are equivalent.

"$\Rightarrow$" : If $g_X \leq 1$, then $\mathrm{Aut}(X)$ is infinite. For instance, consider the case that $X = C_1 \cup C_2$, where $C_1 \cap C_2 = \{p, q\}$ and $C_1 \simeq \mathbb{P}^1 \simeq C_2$, then $g_X = 1$, and the subgroup G of $\mathrm{Aut}(C_1)$ consisting of elements that fix p_1 and p_2 is an infinite group, and it lifts to a subgroup of $\mathrm{Aut}(X)$, hence $\mathrm{Aut}(X)$ is an infinite group. The other cases for $g_X \leq 1$ are similar to this.



If $g_X \geq 2$, to reach a contradiction, assume that there exists an irreducible component $C \subset X$ such that

$$C \simeq \mathbb{P}^1 \quad \text{and} \quad |C \cap (\overline{X \setminus C})| + 2|C_{sing}| \leq 2.$$

Case 1. If $C_{sing} \neq \emptyset$, then we must have $C \cap (\overline{X \setminus C}) = \emptyset$ and $|C_{sing}| = 1$. This implies that $X = C$ and hence $g_X = 1$; however, in this case, $\mathrm{Aut}(X)$ is infinite—a contradiction.

Case 2. If $C_{sing} = \emptyset$, then $|C \cap (\overline{X \setminus C})| \leq 2$. Let $G \subset \mathrm{Aut}(C) = \mathrm{PGL}(2,k)$ be the subgroup that fixes the points in $C \cap (\overline{X \setminus C})$. Then G is an infinite group as it fixes at most two points, and it lifts to a subgroup of $\mathrm{Aut}(X)$. This again implies that $\mathrm{Aut}(X)$ is infinite—a contradiction.

"$\Leftarrow$" : Assume (G_X, w_X) is stable and $g_X \geq 2$. For any $v \in V$, let C_v be the corresponding irreducible component, $B_v := \nu^{-1}\left(C_v \cap (\overline{X \setminus C_v})\right) \bigcup \nu^{-1}(C_{v\,sing}) \subset C_v^\nu$, and

$$\mathrm{Aut}(C_v^\nu, B_v) := \{\alpha_v \in \mathrm{Aut}(C_v) \mid \alpha_v(B_v) = B_v\}.$$

Then, similarly to the argument above for "$\Rightarrow$" when $g_X \leq 1$, by studying the cardinality of B_v and the genus g_{C_v}, and using the fact that a smooth projective curve of genus $g \geq 2$ has finite automorphism group (see [Har77, Chapter IV, Exercise 5.2, Page 348]), we see that $\mathrm{Aut}(C_v^\nu, B_v)$ is a finite group.

Define

$$\mathrm{Aut}^{\#}(X) := \{\alpha \in \mathrm{X} \mid \alpha(C_v) = C_v\}$$

Then, by construction, $\mathrm{Aut}^{\#}(X)$ can be viewed as a subgroup of $\prod_{v \in V} \mathrm{Aut}(C_v^\nu, B_v)$ via the injective morphism

$$\mathrm{Aut}^{\#}(X) \hookrightarrow \prod_{v \in V} \mathrm{Aut}(C_v^\nu, B_v), \quad \alpha \mapsto (\nu^*(\alpha|_{C_v}))_{v \in V}.$$

Therefore, $\mathrm{Aut}^{\#}(X)$ is a finite group.

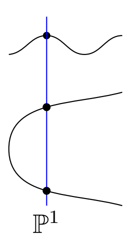
Let $\mathfrak{S}_\gamma$ be the symmetric group (permuting the components), then we have a left short exact sequence

$$0 \to \mathrm{Aut}^{\#}(X) \to \mathrm{Aut}(X) \to \mathfrak{S}_\gamma.$$

Therefore, we see $\mathrm{Aut}(X)$ is a finite group.

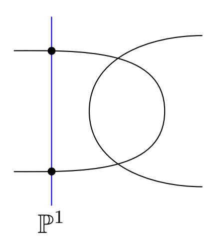
□

Definition 4.11. *A nodal curve X is semi-stable if for any irreducible component $C \subset X$ such that $C \simeq \mathbb{P}^1$, we have $|C \cap (\overline{X \setminus C})| \geq 2$.*



$\mathbb{P}^1$

stable



$\mathbb{P}^1$

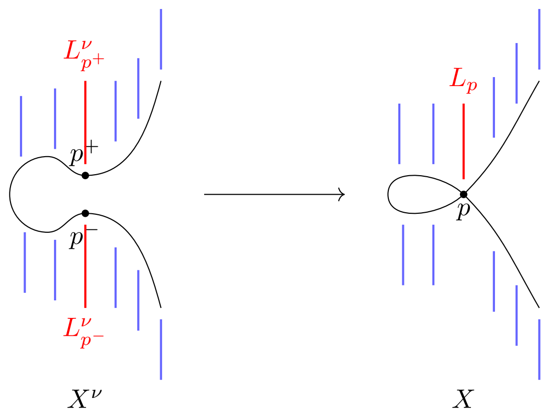
semi-stable but non-stable

4.2 Jacobian of a Nodal Curve

Let $X = \bigcup_{i=1}^{\gamma} C_i$ be a nodal curve, and its normalization map is $\nu\colon X^\nu \to X$, where $X^\nu = \bigsqcup_{i=1}^{\gamma} C_i^\nu$. Then we have

$$\nu^*\colon \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(X^\nu) = \prod_{i=1}^{\gamma} \operatorname{Pic}(C_i^\nu).$$

The map ν^* is surjective. To see this, geometrically, given $L^\nu \in \operatorname{Pic}(X^\nu)$, to construct a line bundle $L \in \operatorname{Pic}(X)$ such that $\nu^* L = L^\nu$, it suffices to identify $L^\nu_{p^-}$ and $L^\nu_{p^+}$ for all $p \in X_{sing}$.



More precisely, consider the following short exact sequence:

$$0 \to \mathcal{O}_X^* \to \nu_* \mathcal{O}_{X^\nu}^* \to \mathcal{S}^* \to 0,$$

taking the long exact sequence associated to it, we obtain

$$0 \to H^0(\mathcal{O}_X^*) \to H^0(\nu_* \mathcal{O}_{X^\nu}^*) \to H^0(\mathcal{S}^*) \to H^1(\mathcal{O}_X^*) \to H^1(\nu_* \mathcal{O}_{X^\nu}^*) \to 0.$$

As $H^1(\mathcal{O}_X^*) = \operatorname{Pic}(X)$ and $H^1(\nu_* \mathcal{O}_{X^\nu}^*) = \operatorname{Pic}(X^\nu)$, we see $\nu^*\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X^\nu)$ is surjective.

Moreover, since $H^0(\mathcal{O}_X^*) = k^*$, $H^0(\nu_* \mathcal{O}_{X^\nu}^*) = (k^*)^\gamma$, $H^0(\mathcal{S}^*) = (k^*)^\delta$, from the long exact sequence, we also have

$$0 \to (k^*)^{b_1(G_X)} \to \operatorname{Pic}(X) \xrightarrow{\nu^*} \operatorname{Pic}(X^\nu) = \prod_{i=1}^{\gamma} \operatorname{Pic}(C_i^\nu) \to 0,$$

as $b_1(G_X) = \delta - \gamma + 1$.

Now we introduce the definition of multi-degree of a line bundle on X.

Definition 4.12. *Let $X = \bigcup_{i=1}^{\gamma} C_i$ be a nodal curve. For any $L \in \operatorname{Pic}(X)$, we have $L|_{C_i} \in \operatorname{Pic}(C_i)$, $\deg_{C_i} L \coloneqq \deg L|_{C_i} = \deg \nu^* L|_{C_i^\nu}$. Then the multi-degree of L is defined as*

$$\underline{\deg} L \coloneqq (\deg_{C_1} L, \cdots, \deg_{C_\gamma} L) \in \mathbb{Z}^\gamma,$$

and the degree of L is defined as $\deg L \coloneqq \sum_{i=1}^{\gamma} \deg_{C_i} L$.

For any $d \in \mathbb{Z}$, the set of line bundles of degree d on X is denoted as

$$\operatorname{Pic}^d(X) := \{L \in \operatorname{Pic}(X) \mid \deg L = d\},$$

similarly, for any $\underline{d} \in \mathbb{Z}^\gamma$, we have

$$\operatorname{Pic}^{\underline{d}}(X) := \{L \in \operatorname{Pic}(X) \mid \underline{\deg} L = \underline{d}\}.$$

With these notations, clearly, we have $\operatorname{Pic}^0(X) = \bigsqcup_{|\underline{d}|=0} \operatorname{Pic}^{\underline{d}}(X)$. Moreover, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (k^*)^{b_1(G_X)} & \longrightarrow & \operatorname{Pic}(X) & \xrightarrow{\nu^*} & \operatorname{Pic}(X^\nu) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
 & & (k^*)^{b_1(G_X)} & \longrightarrow & \operatorname{Pic}^{\underline{d}}(X) & \xrightarrow{\nu^*} & \operatorname{Pic}^{\underline{d}}(X^\nu) & &
\end{array}$$

Definition 4.13. *The generalized Jacobian of a nodal curve $X = \bigcup_{i=1}^{\gamma} C_i$ is defined as*

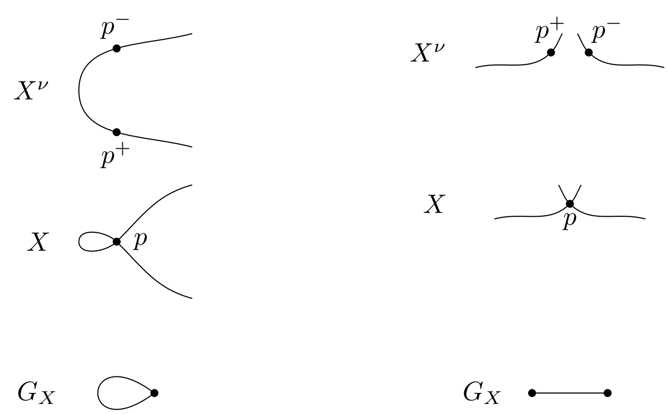
$$\operatorname{Jac}(X) := \operatorname{Pic}^{(0,\cdots,0)}(X).$$

Then we have the following short exact sequence

$$0 \to (k^*)^{b_1(G_X)} \to \operatorname{Jac}(X) \xrightarrow{\nu^*} \operatorname{Jac}(X^\nu) \to 0,$$

where $\operatorname{Jac}(X^\nu) = \operatorname{Pic}^{(0,\cdots,0)}(X^\nu) = \prod_{i=1}^{\gamma} \operatorname{Pic}^0(C_i) = \prod_{i=1}^{\gamma} \operatorname{Pic}^0(C_i^\nu)$ is an abelian variety. Therefore, $\operatorname{Jac}(X)$ is a so-called *semi-abelian variety.*

Example 4.14. *Here are two examples of the generalized Jacobians of nodal curves:*



$0 \to k^* \to \operatorname{Pic}^0(X) \to \operatorname{Pic}^0(X^\nu) \to 0 \qquad 0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}^0(X^\nu) \to 0$

Lemma 4.15. *Let X be a nodal curve of genus $g \geq 0$. Then* $\dim \operatorname{Jac}(X) = g$. *Moreover, the following are equivalent:*

(a) $\mathrm{Jac}(X)$ *is projective;*

(b) $\mathrm{Jac}(X) \simeq \mathrm{Jac}(X^\nu)$*;*

(c) G_X *is a tree, i.e.,* $b_1(G_X) = 0$.

Proof. Exercise. □

Definition 4.16. *If the equivalent conditions in Lemma 4.15 are satisfied, we say the nodal curve* X *is of compact type.*

We use K_{X^ν} to denote the canonical line bundle on X^ν, that is, $K_{X^\nu}|_{C_i^\nu} = K_{C_i^\nu}$ for any component C_i.

Fact 4.17. *We have* $\nu^*\omega_X = K_{X^\nu}\left(\sum_{p\in X_{sing}}(p^+ + p^-)\right)$. *See [ACG11, Pages 91 and 101] for explicit descriptions and discussions.*

References

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